Generalized functions - HW 2

November 10, 2015

Question 1

We first prove a small auxiliary claim. A topological vector space is a regular space.
To show this claim it will be enough to show that for any closed set A such that $0 \notin A$ there exists two disjoint open sets U_1, U_2 such that $0 \in U_1$, $A \subset U_2$.

Indeed, let A be such a set and denote by W its open complement. Then W is a neighborhood of 0 and by exercise 2 there exists a balanced open set $U \subset W$, and we may also assume that $U + U \subset W$.

Now, define $U_1 = U$ and $U_2 = A + U$. Clearly U_1 is open, and $U_2 = \bigcup_{a \in A} U + a$ is open as well. To see that $U_1 \cap U_2 = \emptyset$ consider $x \in U_1 \cap U_2$, then $z = u_1$ and $z = a + u_2$ for some $a \in A$, $u_1, u_2 \in U$. But then $a = u_1 - u_2 \in W - W \subset U = A^c$, contradiction.

So, by Urysohn's metrization theorem (ii) $\implies$ (i), while (i) $\implies$ (ii) trivially. Since each open neighborhood of 0 contains an open balanced neighborhood of 0, and there is a correspondence between open balanced neighborhood of 0 to semi-norms (ii) $\implies$ (iii). Finally, a countable collection of semi-norms implies a countable basis of open balanced sets around 0 and (iii) $\implies$ (ii).

Question 2

Let $0 \in U$ be open, since scalar multiplication is continuous there exists $\varepsilon > 0$ such that for every bounded open $V \subset U$ we have $D_\varepsilon V \subset U$, where $D_\varepsilon = \{x \in F||x| < \varepsilon\}$. Otherwise we could define the sequence $\{v_n\}$ to be elements of V for which $\frac{1}{n}v_n \notin U$ which is a contradiction. Clearly $D_\varepsilon V$ is balanced and the claim follows.

Question 3

Homogeneity and positivity are immediate from the definition of N_C. To see the triangle inequality, let u and v be two vectors such that $u \in \alpha_1 C$ and $v \in \alpha_2 C$ for $\alpha_1, alpha_2 \in F^+$. Since C is balanced it holds that $\frac{\alpha_i}{\alpha_1+\alpha_2}C \subset C$ and convexity of C yields $\frac{u}{\alpha_1+\alpha_2}v + \frac{v}{\alpha_1+\alpha_2}u \in C$. Thus, $u + v \in (\alpha_1 + \alpha_2)C$ and $N_C(u + v) \le \alpha_1 + \alpha_2$. Since this is true for any such α_1, α_2 it is also true for their infimums which concludes the proof.

Question 4

We first observe that if a topological vector space admits a continuous norm then the open convex set define as its unit ball does not contain a line. So, it will be enough to find a locally convex topological vector space such that every non empty open set contains a line.

Consider $\mathbb{R}^\omega$ with the product topology. It is Hausdorf as a product of Hausdorf spaces, and addition and multiplication by scalar are continuous since the projections are continuous. By the definition of the product topology, for any open $\emptyset \neq U \in \mathbb{R}^\omega$ there exists j such that if π_j is the projection on the j'th coordinate then $\pi_j(U) = \mathbb{R}$. In particular if $a \in U$ then $\{x \in \mathbb{R}^\omega | \pi_i(x) = \pi_i(a), i \neq j\}$ is a line which is contained in U.

Question 5

a. Let $\varphi \in W^*$. Zorn's lemma allows us to consider a basis for W, $(v_i)_{i\in I}$ and the extend it to a basis for V, $(v_i)_{i\in I'}$ where $I \subset I'$. We now define $\bar{\varphi} \in V^*$ on the basis in the following way: $\bar{\varphi}(v_i) = \varphi(v_i)$ if $i \in I$ and 0 otherwise. Clearly, $\bar{\varphi} \restriction_W = \varphi$ and the restriction map is onto.

b. Was solved in the tirgul (Exercise 2.12.).

Question 6

As seen in class, we may define the topology on $C^\infty(\mathbb{R})$ with the semi-norms $\|\cdot\|_{n,k}$ defined as $\|f\|_{n,k} = \sup\{|f^{(n)}(x)| | x \in [-k,k]\}$. Thus, it is a locally convex space, and since it is defined by a countable set of norms it is also first countable (by question 1). So, completeness is equivalent to sequential completeness in this case. It will suffice to show that if (f_m) is a Cauchy sequence then (f_m) converges to some $f \in C^\infty(\mathbb{R})$.

Let (f_m) be such a Cauchy sequence. So, for every $n,k \in \mathbb{N}$, (f_m) is Cauchy with respect to $\|\cdot\|_{n,k}$. So, on each compact K all the derivatives of f form a Cauchy sequence with respect to the uniform norm and thus each $(f_m^{(n)})$ converges uniformly to some f_n. A popular theorem from calculus 2 (or 1, depends where you're from) shows us that f_n must in fact equal $f_0^{(n)}$ and $(f_m) \to f_0$ inside K. The claim then follows by compact exhaustion of $\mathbb{R}$.

Question 7

We have seen in class that the topology on $C_c^\infty(\mathbb{R})$ is the co-limit topology of $\lim\limits_{K\subset\mathbb{R}} C^\infty(K)$, K is compact. So $U \subset C_c^\infty(\mathbb{R})$ is open iff its 'restriction' to $C^\infty(K)$ is open for every compact K.

Now, suppose $(f_n) \to f$ in this topology, W.L.O.G. we may assume supp$(f) \subset [-1,1]$ (we can always re-parametrize f). We will show that $\bigcup\limits_n \text{supp}(f_n)$ is compact. Otherwise, for each $k \in \mathbb{N}$ there exists $x_k \notin [-k,k]$ and f_{n_k} such that $|f_{n_k}(x_k)| = \alpha_k > 0$. If we define the open set $U = \{g : |g(x_k)| < \alpha_k, \forall k \in \mathbb{N}\}$ then clearly $f \in U$ but for any $N \in \mathbb{N}$ there exists $n_k > N$ such that $f_{n_k} \notin U$, thus $f_n \nrightarrow f$. To see that U is open, its enough to consider its restriction to each compact $C^\infty(K)$. But then we only have a finitely many x_k in K. And

$U \cap C^\infty(K) = \bigcap_{k: x_k \in K} \{g : |g(x_k)| < \alpha_k\} = \bigcap_{k: x_k \in K} \delta_{x_k}^{-1}(-\alpha_k, \alpha_k)$. Since δ_{x_k} is a continuous function, the pre-image of an open set is open.

So, if $K_f = \bigcup_n \text{supp}(f_n)$ is compact we know that f_n converges to f inside K_f which implies uniform convergence of all derivatives inside K_f and thus in $\mathbb{R}$.

For the other direction, suppose $\bigcup_n \text{supp}(f_n) \cup \text{supp}(f) = K_f$ is compact and that all derivatives of f_n converge uniformly to f. Let $f \in U$ be an open set. So U restricted to $C^\infty(K_f)$ is open. Then, by definition $f_n \to f$ inside $C^\infty(K_f)$. Thus, for almost all n, $f_n \in U \cap C^\infty(K_f) \subset U$ and $f_n \to n$ in the topology of $C_c^\infty(\mathbb{R})$ as desired.